THE REFLECTION OF PLANE NONSTATIONARY WAVES FROM AN ARBITRARY INHOMOGENEOUS HALF-SPACE. THE ACOUSTIC CASE

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 63–67, 1965

It is shown that the acoustics problem of the propagation of plane waves "incident" at a fixed angle to inhomogeous half-space ("two-dimensional" problem) is equivalent to the case of normal incidence to some "fictitious" half-space with the corresponding velocity $v_2(z) = v_1(z)/\cos\alpha(z)$ ("one-dimensional" problem).

Algorithms, by means of which approximate (taking into account only a definite number of "secondary" waves) and exact (taking into account all possible "secondary" waves) calculations of the wave fields can be made, are given.

The problem of the propagation of a plane nonstationary wave incident at some fixed angle α_0 to a half-space whose parameters are arbitrary functions of only one space coordinate x is examined.

It is shown that fundamentally this problem differs in no way from the case of normal incidence of plane waves to a half-space. In other words, the "two-dimensional" problem of plane-wave propagation in acoustics reduces to the "one-dimensional" problem, whose solution was given in [1].

Let the x axis be the interface of two half-spaces 1 and 2, the wave processes in which are described by a single wave equation (acoustic case); the upper half-space 1 is homogeneous and the lower half-space 2 is inhomogeneous (Fig. 1).

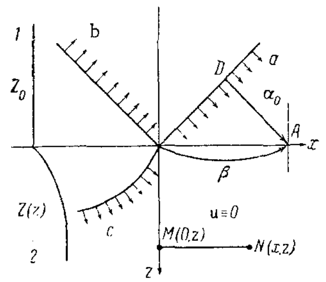

Fig. 1

Just as in [1], we shall assume that the propagation velocity of a disturbance v and the density of the medium ρ are arbitrary functions of depth such that: (a) the wave impedance

$$Z = v(z)\,\rho(z) \qquad (1)$$

remains continuous; and (b) the derivative of the wave impedance with respect to the coordinate z remains bounded in the entire inhomogeneous half-space

$$\left|\frac{dZ(z)}{dz}\right| < N. \qquad (2)$$

Later on, constraints (a) and (b) will be eliminated, since the constructed solution permits extension to the case of discontinuous media.

Let a generator of plane waves, which are propagated at some fixed angle α_0 to the inhomogeneous half-space, be located in medium 1. The process of propagation of plane nonstationary waves in such a medium will be understood as a solution of the equations of acoustics satisfying the following conditions. At any fixed moment in time t_0, there are three wave fronts in the medium: (a) the plane front of the incident wave; (b) the plane front of the reflected wave; and (c) the curvilinear front of the refracted wave; in the domain of (a)-(c), which is a function of t_0, the displacement of the points U(x, z, t) = 0.

An arbitrary function $f(t)$, which describes the "initial" displacement of the points of the homogeneous half-space in time, is specified in the vicinity of the incident-wave front.

With the given $f(t)$, it is required to determine the displacement of any point of the medium such that conditions (a), (b), and (c) are satisfied.

The introduction of this idealization of the process of plane-wave propagation is based in the following premises.

1) Each ray of the plane incident wave satisfies the Fermat principle; in other words, along any ray

$$\sin\alpha(z)/v(z) = \text{const} = \sin\alpha_0/v_0. \qquad (3)$$

2) It is easy to show that the travel time along a ray βA, which satisfies (3), is not less than that along the straight ray DA. In other words, disturbances that are propagated along rays that satisfy (3) cannot overtake disturbances that are propagated along rays of direct waves for any point of the inhomogeneous medium and for any law of change in velocity v(z).

3) Since the parameters of the medium are functions only of z, it can be asserted that the process of "plane wave" propagation in such a medium is "self-similar," with velocity $v^* = v_0 \csc\alpha_0$ along the x aixis, i.e., if the displacements U(t, x, z) at two points M(0, z) and N(x, z) lying on a single horizontal are examined, then they are identical, if we do not take into account the time lag

$$t(x) = (x/v_0)\sin\alpha_0 \qquad (4)$$

of the beginning of oscillations at one point relative to the other. In other words, the displacements of points of the medium U(x, z, t) is a function of not three, but two independent variables, i.e.,

$$\mathbf{U}(t, x, z) = \mathbf{U}(\xi, z) \qquad (\xi = t - (x/v_0)\sin\alpha_0). \qquad (5)$$

Conditions (5) actually indicate that the "two-dimensional" problem of acoustics degenerates to the "one-dimensional" in the case of incidence of plane nonstationary waves.

In order to construct such a solution, it is necessary to solve a system of partial differential equations in the displacement

$$\begin{gathered}\frac{\partial}{\partial x}\left\{\rho(z)\,v^2(z)\left[\frac{\partial U_x}{\partial x} + \frac{\partial U_z}{\partial z}\right]\right\} = \rho(z)\,\frac{\partial^2 U_x}{\partial t^2}\\ \frac{\partial}{\partial z}\left\{\rho(z)\,v^2(z)\left[\frac{\partial U_x}{\partial x} + \frac{\partial U_z}{\partial z}\right]\right\} = \rho(z)\,\frac{\partial^2 U_z}{\partial t^2}.\end{gathered} \qquad (6)$$

In the case of incident plane waves, the Cauchy problem for (6) is difficult to formulate, since at time t = 0 not all of the wave field, but only a part of it (at the front of the incident plane wave a) is assumed to be known. In order to avoid this difficulty, we will use the new variables ξ and τ. The variable ξ (time analog) is defined in (5); the variable τ (analog of depth z) is defined by

$$\tau = \int_0^z \frac{\cos\alpha(z)}{v(z)}\,dz, \qquad (7)$$

where α(z) has the meaning of the angle of incidence and satisfies (3). On the strength of (5) and (7), we can write

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial z} \to \frac{\cos\alpha(\tau)}{v(\tau)}\frac{\partial}{\partial \tau}, \qquad (8)$$

$$\frac{\partial}{\partial x} \to -\frac{\sin\alpha_0}{v_0}\frac{\partial}{\partial \xi} = -\frac{\sin\alpha(\tau)}{v(\tau)}\frac{\partial}{\partial \xi}, \qquad (8) \text{ (cont'd)}$$

and system (6) takes the form

$$\begin{gathered}\frac{\partial}{\partial \xi}\Phi(\xi,\tau) = 0\\ \frac{\partial^2 U_z(\xi,\tau)}{\partial\tau^2} + 2p'(\tau)\frac{\partial U_z(\xi,\tau)}{\partial\tau} = \frac{\partial^2 U_z(\xi,\tau)}{\partial\xi^2} + \\ + \sin\alpha(\tau)\left[\frac{v'(\tau)}{v(\tau)} + \frac{\rho'(\tau)}{\rho(\tau)}\right]\Phi(\xi,\tau) + \\ + \cos\alpha(\tau)\sin\alpha(\tau)\frac{\partial}{\partial\tau}\frac{\Phi(\xi,\tau)}{\cos\alpha(\tau)} = 0,\end{gathered} \qquad (9)$$

where

$$\begin{gathered}\Phi(\tau,\xi) \equiv \sin\alpha(\tau)\frac{U_z(\xi,\tau)}{\partial\tau} + \cos\alpha(\tau)\frac{\partial U_x(\xi,\tau)}{\partial\xi}\\ p(\tau) \equiv \frac{1}{2}\ln\frac{v(\tau)\rho(\tau)}{\cos\alpha(\tau)}\frac{\cos\alpha_0}{v_0\rho_0}, \quad p(\tau) \equiv 0 \text{ for } \tau < 0,\end{gathered} \qquad (10)$$

and the prime indicates the derivative with respect to τ.

It is how possible to formulate the Cauchy problem for system (9): to find a solution of system (9) with the initial data

$$U_z(\xi,\tau)|_{\xi\leqslant 0} = \delta(\xi-\tau), \qquad U_x(\xi,\tau)|_{\xi\leqslant 0} = \operatorname{tg}\alpha_0\delta(\xi-\tau). \qquad (11)$$

Here, δ^* is the Dirac delta-function symbol.

Thus, system (6) in the class of plane waves corresponds to system (9) with initial data of the type (11), which, in turn, admits reduction of order.

We shall show that system (9) with intial data (11) is equivalent to the system

$$\begin{gathered}\sin\alpha(\tau)\frac{\partial U_z(\xi,\tau)}{\partial\tau} + \cos\alpha(\tau)\frac{\partial U_x(\xi,\tau)}{\partial\xi} = 0\\ \sin\alpha(\tau)\frac{\partial U_z(\xi,\tau)}{\partial\xi} + \cos\alpha(\tau)\frac{\partial U_x(\xi,\tau)}{\partial\tau} + \\ + \frac{\rho'(\tau)}{\rho(\tau)}\cos\alpha(\tau)\,U_x(\xi,\tau) = 0\end{gathered} \qquad (12)$$

with the same initial conditions (11).

In fact, the left side of the first equation of system (12) agrees with the first equation of (10). The first equation of (9), therefore, is automatically satisfied, and the second equation transforms to

$$\frac{\partial^2 U_z(\xi,\tau)}{\partial\tau^2} + 2p'(\tau)\frac{\partial U_z(\xi,\tau)}{\partial\tau} = \frac{\partial^2 U_z(\xi,\tau)}{\partial\xi^2}. \qquad (13)$$

If the second equation of (12) is differentiated with respect to ξ and the the first equation of (12) is used, we arrive at the relation

$$\sin\alpha(\tau)\left[\frac{\partial^2 U_z}{\partial\xi^2} - \frac{\partial^2 U_z}{\partial\tau^2} - 2p'(\tau)\frac{\partial^2 U_z}{\partial\tau}\right] = 0. \qquad (14)$$

Inasmuch as $\sin\alpha(\tau) \neq 0$, since "oblique" incidence is being considered, Eq. (14) concides with Eq. (13). Thus, any solution $U(\xi,\tau)$ of system (12) is contained among the solutions $W(\xi,\tau)$ of system (9). Let us set up the difference $\varepsilon(\xi,\tau) = W(\xi,\tau) - U(\xi,\tau)$, which, obviously, will also be a solution of system (9), but now with zero initial conditions, since the initial data for W and U are one and the same. Hence, $\varepsilon(\xi,\tau) \equiv 0$, from which follows the assertion of the equivalance of systems (9) and (12) for plane waves, i.e., for one and the same initial conditions of type (11).

By similar reasoning it can be shown that system (12) with initial conditions (11) is equivalent to Eq. (13) (for determination of the vertical component of the displacement) with the initial data

$$U_z(\xi,\tau)|_{\xi\leqslant 0} = \delta(\xi-\tau), \qquad (15)$$

which exactly coincides with (8) from [1].

If we change from τ to the coordinate z in Eq. (13), in accordance with (7) and (8), then we obtain

$$\frac{\partial}{\partial z}\left[\rho(z)\frac{v^2(z)}{\cos^2\alpha(z)}\frac{\partial U_z(\xi,z)}{\partial z}\right] = \rho(z)\frac{\partial^2 U_z(\xi,z)}{\partial\xi^2}. \qquad (16)$$

By comparing (16) with (15) from [1], we can see that the "two-dimensional" problem in acoustics for plane waves is equivalent to the "one-dimensional" problem, in which the role of the velocity is played by the expression $v(z)/\cos\alpha(z)$ and the role of time is played by the variable ξ, which was defined in (5).

It was shown in [1] that for one and the same initial data (15), Eq. (13) is equivalent either to a system of integral relations of the form

$$\begin{gathered}\Psi_1(\xi_0,\tau) = -\int_\tau^{\xi_0}\Psi_2(\xi_0,x)\,dp(x)\\ \Psi_2(\xi_0,\tau) = \delta(2\xi_0 - 2\tau) + \int_0^\tau \Psi_1(\xi_0 - \tau + x, x)\,dp(x)\end{gathered} \qquad (17)$$

or to a system of differential equations in first-order partial derivatives

$$\begin{gathered}\frac{\partial\Psi_1(\xi_0,\tau)}{\partial\xi} = \frac{\partial\Psi_1(\xi_0,\tau)}{\partial\tau} - p'(\tau)\Psi_2(\xi_0,\tau)\\ \frac{\partial\Psi_2(\xi_0,\tau)}{\partial\xi} = -\frac{\partial\Psi_2(\xi_0,\tau)}{\partial\tau} + p'(\tau)\Psi_1(\xi_0,\tau)\end{gathered} \qquad (18)$$

where $p(\tau)$ is defined in (10); the quantity ξ_0 has the form

$$\xi_0 = {}^1/_2(\xi+\tau). \qquad (19)$$

The vertical component of the displacement is determined in this case by the formula

$$U_z(\xi,\tau) = [\Psi_1(\xi_0,\tau) + \Psi_2(\xi_0,\tau)]\,e^{-p(\tau)}. \qquad (20)$$

The horizontal component, as is easily shown, can be found from the formula

$$U_x(\xi,\tau) = [\Psi_2(\xi_0,\tau) - \Psi_1(\xi_0,\tau)]\,e^{-p(\tau)}\operatorname{tg}\alpha(\tau). \qquad (21)$$

From system of integral relations (17), it follows that when $|p'(x)| < \infty$, a solution always exist.[1] In addition, the solution satisfies conditions (a)-(c) given at the beginning of the article, and when $\xi_0 < \tau$ it vanishes identically. In other words, the condition $\xi_0 <$ which, using (19), (5), and (7), can be written as

$$t - \frac{x}{v_0}\sin\alpha_0 - \int_0^z \frac{\cos\alpha(z)}{v(z)}dz < 0, \qquad (22)$$

determines the refracted wave front. In a homogeneous medium, as can easily be seen, inequality (22) is a condition for determining the front of an ordinary plane wave.

Thus, in order to find the solution of system (6) in the class of plane waves, it is necessary to solve: a) either system of differential equations (12) or (18); b) system of integral relations (17); or c) differential equation (13) or (16) with the corresponding initial conditions of type (11) and (15), and to use relations (20) and (21).

Note that a system similar to (18) was obtained in [2] for a more general problem: the incidence of a plane nonstationary wave on an elastic half-space. In that paper, however, there is no system of integral relations like (17), so that the author of [2] was unable to make estimates in the method of successive approximations. Nor is these an extension to the case of discontinuous media, or algorithms for calculating the total wave field taking into account "secondary" waves. Let us consider these problems.

[1] A proof of this statement is given in [1].

The points at which the function $|p'(x)|$ is not bounded will be singular points of integral relations (17). This unboundedness may be caused by two factors: a) a discontinuity of the wave impedance[1] $Z = \rho(x)v(x)$, if condition (2) is abandoned; and b) the presence of a "turning" point of the ray, at which $\cos\alpha(x) = 0$. From a formal point of view, these two cases are the same.

In order to extend the solution to the case of unbounded $|p'(x)|$, let us examine an inhomogeneous layer between two homogeneous half-spaces, which degenerates to a sharp interface (i.e., the thickness of the layer $H \to 0$ with preservation of the wave impedances outside of the layer; therefore, $|p'(x)| \to \infty$).

In this case, all "secondary" waves in the layer "gather" at a single point at the same moment of time. Therefore, we must pass from $U_z(\xi, \tau)$ to the total integral amplitude, defined by the expression

$$S_z = \int_0^\infty U_z(\xi, \tau)\, d\xi, \tag{23}$$

where, from practical considerations, S_z must be finite [1]. As in [1], in the case $|p'(x)| = \infty$ the solution can be formally determined by letting

$$U_z(\xi_1) = -\frac{\chi(\tau_2) - \chi(\tau_1)}{\chi(\tau_2) + \chi(\tau_1)}\,\delta(\xi - \tau_1) \tag{24}$$

for the upper half-space and

$$U_z(\xi_2) = \frac{2\chi(\tau_1)}{\chi(\tau_2) + \chi(\tau_1)}\,\delta(\xi - \tau_2) \tag{25}$$

for the lower half-space, if the initial conditions were assigned in the form (15).

In expressions (24) and (25), the normal impedance of the medium is defined as [3]

$$\chi(\tau_i) = \rho(\tau_i)\, v(\tau_i) / \cos\alpha(\tau_i) \qquad (i = 1, 2), \tag{26}$$

where τ_1 and τ_2 are certain points of the upper and lower homogeneous half-space, respectively. The thus-defined functions $U_z(\xi_1)$ and $U_z(\xi_2)$ may be considered new initial impulses for the homogeneous half-spaces.

Note that formulas (24) and (25) can be extended, as in [3], to the case of complex impedances of the medium, i.e., they can be used in the geometric-shadow region, where $\chi(\xi_1)$ is a purely imaginary quanitity.

Now let us turn to the problem of approximate and exact calculation of the wave field. For definiteness, we shall consider a reflected wave, i.e., $\tau = 0$.

In many cases of practical interest, it is often sufficient to examine not the entire wave field, but simply waves that have undergone one reflection in the layer [1]. This is explained as follows.

As is known, a reflected wave $\varphi(t)$, which corresponds to some arbitrary continuous function $f(t)$ at the incident-wave front, is defined as

$$\varphi(t) = \int_0^t U_\delta(\xi)\, f(t - \xi)\, d\xi. \tag{27}$$

Here, $U_\delta(\xi)$ is the solution in the case of initial data (11) and (15).

If the function $|p'(x)|$ is piecewise-continuous, then $U_\delta(\xi)$ can always be divided into two parts: a discontinuous part $U_1(\xi)$, which corresponds to waves that have undergone one reflection in the layer, and a continuous part $U_2(\xi)$, which corresponds to all other "secondary" waves [1]. Then

$$\varphi(t) = \int_0^t U_1(\xi)\, f(t - \xi)\, d\xi + \int_0^t U_2(\xi)\, f(t - \xi)\, d\xi. \tag{28}$$

If $f(t)$ is a strongly oscillating function, as compared with $U_2(\xi)$, then the second term of the right side of (28) gives a certain background, which, in many cases cannot be taken into account, and the wave field is calculated from the formula

$$U(t) = -\frac{1}{2}\int_0^t p'\left(\frac{\xi}{2}\right) f(t - \xi)\, d\xi. \tag{29}$$

For each specific case of assignment of the parameters of the medium and the initial impulse $f(t)$, by using the formula in [1] we can make calculations that permit accurate determination of the influence of the second term. However, it may not be advisable to do this, since for system (18) we can write a difference scheme, which by solving we get the entire wave field, taking into account all possible "secondary" waves. We can then compare the exact calculation with the approximate one by formula (29). Such a comparison was made and was found to be fully satisfactory for many practical problems.

The difference scheme for (18) is the same as that in [1]. We set $\xi = k\Delta\xi$ and $\tau = n\Delta\tau$; it is assumed that $\Delta\xi = \Delta\tau$. The difference scheme, with account for extension to the case of discontinuous media (24) and (25), can be written as

$$\begin{gathered} U_1(k+1, n) = (1 + q_n)\, U_1(k, n+1) - q_n U_2(k, n) \\ U_2(k+1, n) = (1 - q_{n-1})\, U_2(k, n-1) + q_{n-1} U_1(k, n) \\ \left(q_n = \frac{\rho_{n+1}v_{n+1} / \cos\alpha(n+1) - \rho_n v_n / \cos\alpha(n)}{\rho_{n+1}v_{n+1} / \cos\alpha(n+1) + \rho_n v_n / \cos\alpha(n)} = \right. \\ \left. = \frac{\chi(n+1) - \chi(n)}{\chi(n+1) + \chi(n)}\right). \end{gathered} \tag{30}$$

The displacement components U_z and U_x are given by

$$\begin{gathered} U_z(k, n) = U_1(k, n) + U_2(k, n), \\ U_x(k, n) = \operatorname{tg}\alpha(n)\, [U_2(k, n) - U_1(k, n)]. \end{gathered} \tag{31}$$

It is easy to show that difference scheme (30) approximates (18) with accuracy $O(\Delta\tau)$, converges, and is stable.

Note that (30) permits extension to the case of complex values of q_n, $\Delta\tau$, U_1, and U_2; in other words, we can calculate by (30) in the geometric-shadow region as well.

In conclusion, it would appear desirable to study the solution of the wave equation in such a medium. The problem is to find the solution of the wave equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial Z^2} = \frac{1}{v^2(z)}\frac{\partial^2 U}{\partial t^2} \tag{32}$$

when a plane nonstationary wave is incident to the inhomogeneous half-space at some angle α_0. Reasoning as at the beginning of the article and using (5), (7), and (8), we arrive at the following Cauchy problem:

$$\begin{gathered} \frac{\partial^2 U(\xi, \tau)}{\partial \tau^2} + 2p'(\tau)\frac{\partial U(\xi, \tau)}{\partial \tau} = \\ = \frac{\partial^2 U(\xi, \tau)}{\partial \xi^2} \quad \left(p(\tau) \equiv \frac{1}{2}\ln\frac{v(\tau)}{\cos\alpha(\tau)}\frac{\cos\alpha_0}{v_0}\right) \end{gathered} \tag{33}$$

with the initial conditions

$$U(\xi, \tau)\big|_{\xi \leqslant 0} = \delta(\xi - \tau). \tag{34}$$

Comparing (33) and (34) and (13) and (15), we see that they are actually identical. The difference is that in Problem (33) and (34),

[1] This refers to causes when the wave impedance is continuous, while its derivative is unbounded.

the parameter $p(\tau)$ is somewhat different; it differs from (10) in sign and by the fact that $\rho(\tau) = \text{const}$.

In other words, everything that pertained to $U_z(\xi, \tau)$ to an identical degree to $U(\xi, \tau)$ with the correction to $p(\tau)$ mentioned above.

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